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NOTE ON THE CALCULATION OF RESISTANCE AND FLEXURE OF SOLID STRUCTURAL PARTS WITH SINGLE AND DOUBLE CURVATURE, WITH CONSIDERATION OF THE VARIOUS FORCES ACTING ON THE PARTS IN ALL DIRECTIONS

Barré de Saint-Venant

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NOTE ON THE CALCULATION OF RESISTANCE AND FLEXURE OF SOLID STRUCTURAL PARTS WITH SINGLE AND DOUBLE CURVATURE, WITH CONSIDERATION OF THE VARIOUS FORCES ACTING ON THE PARTS IN ALL DIRECTIONS **/942

Barré de Saint-Venant*

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The theories of Navier, covering only the case of stress in single-curvature solid structural members is rectified and extended to double-curvature parts subjected to simultaneous torsion and flexure, including interaction of the individual fibers and cross sections. Equations of equilibrium for external and internal forces, with emphasis on dilatation and slip, are given indicating the numerical maximum for the various forces in members with free and clamped end. A general method is developed for determining unknown reactions and interactions in any system of solid bodies, which subdivides the member into individual elements and thus defines the forces involved by no more than six indeterminates, eliminating the Poisson transcendental discontinuity formulas.

AUTHOR

Section 1. Introduction

1. By means of expressing the equilibrium of external forces acting on a solid body by the internal forces that manifest themselves across one of its transverse sections it is possible to determine the magnitude of dilatations and contractions undergone by the various parts of the body; on the one hand, this

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* Author's excerpt; commissaries: Cauchy, Poncelet, Piobert, Lamé.

** Numbers in the margin indicate pagination in the original foreign text.

permits defining the conditions of the resistance of the body to given forces and, on the other hand, to calculate - if required - the displacements of its points and the deformations resulting from this.

However, it is known that the equilibrium of forces in space is generally expressed by six equations, three of which are equations of components and three of moments. Despite this fact, the present theory on the resistance of solid bodies, represented by the highly useful and important work done by Navier, /943 never uses more than two equations.

Could this be due to the fact that this theory is always limited to cases for which the other four equations do not exist? This question must be answered in the negative since not only does this theory disregard (for example) the case of double-curvature curves, the case of a body which simultaneously is bent and twisted, etc. but it also disregards, in the treated cases, some of the most basic circumstances.

In addition, this particular theory assumes that all plane sections remain plane and that the individual fibers into which the body is imagined to be subdivided behave as though they were isolated or without mutual interaction. However, recent research, based on exactly this work by Navier and confirmed in experiments by Savart and Cagniard de Latour, makes it quite impossible to accept these two hypotheses in quite a few cases.

Another objection to this theory is its - at least apparently - complexity since it always gives the computation of displacement of the points before the conditions of nonrupture; however, such computation is useless in most of the cases, making it preferable to simply derive the equations of resistance that are of greatest importance for practical use.

Finally, the theory does not furnish a general method for determining the reactions of fixed points or the unknown mutual interaction of various parts of the same system; from this it follows that Navier, while resolving in an entirely satisfactory manner many cases which previously had remained unsolved, finally returned - for many other cases - to purely hypothetical resolutions of the forces which had been entirely adequate before Navier's time.

In my report I am making an attempt to fill these voids, to correct the inaccuracies, and to eliminate any useless complication. I also included in the calculation the effects of lateral slip due to these transverse components, whose omission had been the main point in the objection made by Vicat to the entire theory of the resistance of solids. I will demonstrate that, by means of a second equation of transverse moments, this general case mentioned by Persy can be solved quite simply, where the conventionally posed equilibrium no longer exists and where the flexure of the body occurs necessarily in a direction different from that at which it had been subject to bending. I will also extend the resistance calculations to cases of simultaneous flexure and torsion which are frequently encountered when taking into consideration that a twisted specimen has almost never suffered this torsion by a so-called couple. I will also take into consideration that the plane sections become skewed and that they incline slightly with respect to the central fibers which latter exert a mutual interaction which cannot be neglected. I will also furnish new differential equations for minor displacements of specimens with double curvature and set /944 up extremely simple integrals derived from these three simultaneous equations of the third order with nonconstant coefficients.

In addition, practical application examples will be given for most of the new formulas, followed by a general method for determining the reactions and interactions which cannot be derived from the given forces by ordinary equations of statics: I believe that if a certain effort is made to establish and solve all these equations, although they are numerous but all of the first degree, which result from this method in all cases, then the expression for the conditions of resistance in any framework system will contain just as few indeterminate and arbitrary quantities as those referring to suspension bridges*.

Section 2. Equations of Equilibrium for Interior and Exterior Forces

2. By the term dilatation, we will understand here, in the direction of a short material straight line at the interior of the body, the degree of elongation (positive or negative) suffered by this straight line due to displacements of the molecules. The term slip, on a small plane material surface, is to mean the inclination assumed by a straight line to this surface which, originally, had been perpendicular: The motion of the surface contributes to the slip as does the motion of the straight line.

Let ω denote one of the transverse sections of a solid body, normal to the straight axis or curve that connects their centers of gravity.

Let u and ν be the coordinates of the center m of the element $d\omega$ with respect to the two principal inertia axes of the section passing through the center of gravity M.

Let r be the distance Mm.

Let ∂ be the longitudinal dilatation suffered by one fiber or by a prismatic portion of the body, almost parallel to the axis, having $d\omega$ as base and terminating at a second close-by section ω'.

* A portion of the formulas and methods in this report were furnished in 1837 and 1838 to students of the Mechanical Engineering College (bridges and highways) in the course of applied mechanics which I conducted on suggestion of Mr. Coriolis.

Let g be the slip on the section ω at the point m.

Let g', g'' be this same slip estimated parallel to the two principal axes of u and ν.

Let ∂_0, g_0, g_0', g_0'' be the values of these quantities for $u = 0$, $\nu = 0$. /945

Let $u = \int \nu^2 d\omega$, $u' = \int u^2 d\omega$ be the moments of inertia of the section ω about the axis of u and the axis of ν.

Despite the negligible inclination of the fibers on these two sections, it is quite obvious that, by neglecting the extremely small quantities of higher order as well as the minor influence of the curvature of the sections on the length of the fibers (see below), the expression of this length between the two sections would be of the first degree in u and ν, before as well as after the displacements; the same statement holds for the dilatation. Consequently, the following expression is obtained for the latter:

$$\partial = \partial_0 + au + b\nu. \tag{1}$$

The slip at the point m is due to the following causes: 1) The section ω' has turned through a small angle with respect to the section ω; if θ is the quotient of this small angle by the distance of the two sections, then θr will be the inclination acquired, on the axis of the member, by the previous normal to the section ω at the point m, which leads to projections $\theta\nu$, $-\theta u$ on two planes perpendicular to ω and passing through the axes of u and ν. 2) The axis itself is inclined, on the section, by a small angle g_0 whose projections onto the same planes are denoted by g_0', g_0''. 3) This surface has become skewed. It can be readily demonstrated, on the basis of a few examples, that, if w is the vanishingly short distance of a point on this surface from its central tangential plane, its form must be of the type represented by the equation $w = \gamma u \nu$ (approximately a double windmill sail), which also results from an analysis by Cauchy;

thus, the share of the skewing in the slip, estimated in accordance with the directions of u and ν, can be represented approximately by $-\gamma\nu$, $-\gamma u$.

Consequently,

$$g' = g'_0 + \theta\nu - \gamma\nu, \quad g'' = g''_0 - \theta u - \gamma u.$$

However, the skewing γ and the torsion θ are not independent of each other. Cauchy as well as myself have found, for a rectangular section and by applying his analysis to a section of a different form, that

$$\gamma = \theta . \frac{\mu - \mu'}{\mu + \mu'},$$

from which it follows that

$$g' = g'_0 + \frac{2\mu'}{\mu + \mu'}\theta\nu, \quad g'' = g''_0 - \frac{2\mu}{\mu + \mu'}\theta u. \qquad (2)$$

3. Let now r_ℓ be the sum of the components, parallel to the tangent /946 to the axis of the member at the point M, of all the forces acting from this point up to the extremities of the structural member.

Let P_u, P_ν be the same components in the direction of the principal axes M_u, M_ν of the section.

Let M_ℓ, M_u, M_ν be the sums of the moments of the same forces about the three same rectangular lines.

Let E be the coefficient by which the dilatation of an isolated prism, with a base of 1 m^2, must be multiplied for obtaining the forces able to produce dilatation.

Let G be the coefficient of the same type, with respect to the slip.

Let π_u, π_ν be the lateral pressures to which the fiber under consideration can be subjected on its two faces perpendicular to u and to ν per unit surface. Let us neglect the rare case (as also the case of considerable friction exerted on the sides of the specimen) where these pressures are no longer perpendicular to the fibers.

Let us assume that the force able to produce dilatation of the fiber will be $Ed\omega(\partial_0 + au + b\nu)$ without the pressures π_u, π_v; however, it is known that these two forces each generate a longitudinal dilatation equal to one quarter of the lateral contraction that they are able to produce (in this discussion, I restrict myself to the case of an elasticity equal in all directions); consequently, the longitudinal force will only be

$$[E(\partial_0 + au + b\nu) - \tfrac{1}{4}(\pi_u + \pi_v)]\,d\omega. \tag{3}$$

This force, assuming in first approximation that π_u, π_v are functions of the first degree of u and ν, will then be written as follows:

$$E(\partial'_0 + a'u + b'\nu). \tag{4}$$

The transverse interior forces are obtained on multiplying the expressions (2) by $Gd\omega$.

Thus, using the sums of the components and moments of the interior forces with respect to the same axes as the exterior forces, we will obtain the following expression for the equilibrium, taking into consideration $\int u d\omega = 0$, $\int \nu d\omega = 0$, $\int u\nu d\omega = 0$:

$$\left\{\begin{array}{l} P_l = E\omega\partial'_0, \quad P_u = G\omega g'_0, \quad P_v = G\omega g''_0, \\ M_u = E\mu b', \quad M_v = E\mu' a', \quad M_l = G\cdot\dfrac{2\mu\cdot 2\mu'}{\mu+\mu'}\theta. \end{array}\right. \tag{5}$$

Section 3. Conditions of Resistance to Rupture or to Alteration in Elasticity /947

4. From eqs.(2), (3), (4), (5), we can derive

$$\partial = \frac{P_l}{E\omega} + \frac{M_u}{E\mu}\nu + \frac{M_v}{E\mu'}u + \frac{1}{4}\frac{\pi_u + \pi_v}{E}, \tag{6}$$

$$g' = \frac{P_u}{G\omega} + \frac{M_l}{2G\mu}\nu, \quad g' = \frac{P_v}{G\omega} - \frac{M_l}{2G\mu'}u; \quad \text{resulting in } g = \sqrt{g'^2 + g''^2}. \tag{7}$$

Thus, let $\frac{R_0}{E}$, $\frac{\Gamma_0}{G}$ be the greatest dilatation and the greatest transverse slip to which a prism of the same material can be subjected without risk (where o denotes that the forces R_0, Γ_0 produce no changes in the elasticity, even over prolonged periods of time). Then, the result for the equations of resistance will be as follows:

1) If P_u, P_v, M_ℓ are zero, i.e., if there are only dilatations present (or, properly speaking, flexures produced by the unequal dilatations of the fibers), we have

$$R_0 \overset{=}{\text{or} >} \left| \text{ numerical maximum of } \frac{P_l}{\omega} + \frac{M_u}{\frac{1}{v}\mu} + \frac{M_v}{\frac{1}{u}\mu'} + \frac{1}{4}(\pi_u + \pi_v); \right| \tag{8}$$

2) If P_ℓ, M_u, M_v, π_u, π_v are zero, i.e., if only slip is present (or torsion), we have

$$\Gamma_0 \overset{=}{\text{or} >} \text{ numerical maximum of } \left(\frac{P_u}{\omega} + \frac{M_l}{2.\frac{1}{v}\mu}\right)^2 + \left(\frac{P_v}{\omega} - \frac{M_l}{2.\frac{1}{u}\mu'}\right)^2. \tag{9}$$

5. If both dilatations and slip are present at the same time, these formulas are no longer valid; any slip, in definite oblique directions, produces molecular spreading and crowding, which might render such slip relatively dangerous. This means that the effects of slip coincide with those produced by dilatation.

However, it is easy to prove that, if ∂_g represents the dilatation taking place in the direction of the slip g at the point m under consideration, the following dilatation will be obtained in a direction making an angle φ with the fiber such that $\tan 2\varphi = \frac{g}{\partial - \partial_g}$:

$$\frac{1}{2}(\partial + \partial_g) \pm \frac{1}{2}\sqrt{(\partial - \partial_g)^2 + g^2}, \tag{10}$$

and that this dilatation (positive or negative) is the greatest dilatation that can take place about the point m.

The maximum of eq.(10) must be equated to $\frac{R_0}{E}$ to obtain the resistance /948 equation applicable to all cases. Let us restrict our present calculation to the case in which π_u, π_v are negligible, so that, as is well known, $\partial_g = -\frac{1}{2}\partial$ and eq.(10) becomes*

$$\frac{3}{8}\partial \pm \frac{5}{8}\sqrt{\partial^2 + \left(\frac{4}{5}g\right)^2}. \qquad (11)$$

Taking into consideration, as is also known, $G = \frac{2}{5}E$, we will then have

$$\left\{ R_0 \underset{\text{or}}{\overset{=}{>}} \middle| \begin{array}{l} \text{numerical maximum of } \frac{3}{8}\left(\frac{P_l}{\omega} + \frac{M_u}{\frac{I}{v}\mu} + \frac{M_v}{\frac{I}{u}\mu'}\right) \\ \pm \frac{5}{8}\sqrt{\left(\frac{P_l}{\omega} + \frac{M_u}{\frac{I}{v}\mu} + \frac{M_v}{\frac{I}{u}\mu'}\right)^2 + \left(2\frac{P_u}{\omega} + \frac{M_l}{\frac{I}{v}\mu}\right)^2 + \left(2\frac{P_v}{\omega} - \frac{M_l}{\frac{I}{u}\mu'}\right)^2}. \end{array} \right. \qquad (12)$$

6. This expression reduces to eq.(9) if P_ℓ, M_u, M_v are zero and if it is assumed that $\Gamma_0 = \frac{4}{5} R_0$.

Let us note also that this expression is simpler and more symmetrical than if, with respect to torsion, one would have adhered to the conventional theory which neglects the skewing.

The expression is even further simplified in the most common cases in which the terms in P_ℓ, P_u, P_v are negligible; however, an extraordinary degree of simplicity is obtained when the section becomes a circle, a square, or a four-point star which configuration is frequently used for castings. Then, $\mu = \mu'$ and, if M_d is the greater of the two moments about the diagonals or

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* I discovered this formula in 1837. Poncelet used this formula in his course of industrial mechanics at the faculty and insisted on the necessity to take the implications of this formula into consideration. He was kind enough to cite me in his unpublished reports.

about the larger diameters 2r' of the section, we will have

$$R_o = \text{num. maxim.} \frac{1}{\frac{1}{r'}\mu}\left(\frac{3}{8}M_d \pm \frac{5}{8}\sqrt{M_d^2 + M_l^2}\right); \tag{13}$$

and the quantity of the second term will reduce to $\frac{1}{\frac{1}{4}\pi r'^3}\left(\frac{3}{8}M_d \pm \frac{5}{8}M\right)$ for a circle, where M denotes the total moment of the forces. /949

I would be completely wrong if the simplicity of these resistance formulas for cases of simultaneous flexure and torsion would not prove the validity of the principles used in their derivation.

By eliminating "or >" and "numerical maximum" from these formulas, they will also become so-called equations of equal resistance.

Section 4. Application to Several Examples. Difference from Results of the Earlier Theories

7. Rectangular specimen stressed perpendicularly to its axis but obliquely to the sides of its base. Let b and c be the two sides of the base, a the length of the specimen clamped at one end, P the force stressing this specimen at the other end, and φ the angle made by this force with the side c; we then obtain [eq.(6)]:

$$R_o = \frac{6aP}{b^2c^2}(b\cos\varphi + c\sin\varphi),$$

which is a much simpler formula than the formula of the older theory $\frac{6aP}{bc}\cdot\frac{b\sin\varphi + c\cos\varphi}{b^2\sin^2\varphi + c^2\cos^2\varphi}$ where only a single moment about an oblique straight line is used while the other moment of the interior forces is neglected.

If the angle φ is 45°, the ratio of the values of P derived from the old and from the new formula would be 1.08 at c = 1.5b, or 1.25 at c = 2b, and 1.67 at c = 3b. From this it follows that the old formula may give a false security to the designers, thus inducing them to load the structural parts to more than their load-carrying capacity.

8. Rectangular specimen of short length, clamped at one end and stressed perpendicularly at the other end. In this case, P_u is no longer negligible, yielding

$$R_o = \frac{6aP}{bc^2}\left[\frac{3}{8} + \frac{5}{8}\sqrt{1 + \left(\frac{c}{3a}\right)^2}\right].$$

The value by which the quantity in brackets of the second term exceeds unity represents the portion of the influence of this transverse component which tends to cut the fibers and which had been neglected in the old theory. This excess, for c = a, 2a, 3a, 4a is, successively, $3\frac{1}{2}$, $12\frac{1}{2}$, 26, 42%.

9. Same specimen, with the weight P uniformly distributed over its upper surface. I am using this case as an example for the influence of the lateral pressure on the fibers (No.4). Then, /950

$$\frac{\pi_o}{4} = \frac{P}{4ab},$$

and

$$R_o = \frac{6\frac{a}{2}P}{bc^2}\left[\frac{3}{8}\left(1 + \frac{1}{12}\frac{c^2}{a^2}\right) + \frac{5}{8}\sqrt{\left(1 + \frac{1}{12}\frac{c^2}{a^2}\right)^2 + \frac{1}{9}\frac{c^2}{a^2}}\right].$$

The influence is much greater than that of the transverse slip since the quantity in brackets exceeds unity; for c = a, this excess is 11.5% and, for c = 2a, it is 43%.

Similarly, in the case of the preceding paragraph (No.9), a lateral pressure must be present there, which is difficult to evaluate exactly but whose influence presumably equals that of the slip.

10. Solids of equal resistance. Taking the slip into consideration will eliminate a paradox from the theory of these solids, produced by conventional formulas: It is known that these formulas result in zero thickness at the points of support. My own formulas furnish no such result, presumably due to the fact

that the transverse component P_v had been neglected in the conventional formulas.

11. Rectangular specimen, with simultaneous flexure and torsion. Let us assume that the weight P of the member described in paragraph No.7 acts over the intermediary of a horizontal lever arm of a length h, so that

$$R_{\bullet} = \frac{6aP}{b^2c^2}(b\cos\varphi + c\sin\varphi)^2\left[\frac{3}{8} + \frac{5}{8}\sqrt{1 + \frac{h^2}{a^2}\cdot\frac{b^2+c^2}{(b\cos\varphi + c\sin\varphi)^2}}\right].$$

Let $\varphi = 0$; then, the quantity in brackets will exceed unity,

for $c = b$, $h = \frac{1}{2}a$, by 0.14   for $c = 2b$, $h = \frac{1}{2}a$, by 0.31;

$h = a$, by 0.46 $h = a$, by 0.91.

This constitutes the proportion of the influence of torsion which the new formulas attempt to estimate.

12. Revolving shaft with circular or square cross section, bent and twisted under the action of two gear trains or two drive belts. The very simple formula (13) is used in this case.

13. Horizontal half-circle, clamped or built-in at one end and stressed at the other end by a weight. Here, rupture does not take place at the point of clamping as is the case in straight specimens but at 0.226 of the length, and the resistance is 1.408 times that of a straight specimen having a length /951
equal to its diameter.

14. Vertical spiral spring, extended or compressed by a weight P. Let a be the radius of the cylinder of the axis, r the radius of the spring of circular cross section, and φ the constant angle made by the axis with the horizon so that the complete formula will yield

$$R_{\bullet} = \frac{4P}{\pi r^3}\left[\frac{3}{8}\left(a + \frac{r}{4}\right)\sin\varphi + \frac{5}{8}\sqrt{\left(a + \frac{r}{4}\right)^2\sin^2\varphi + \left(a + \frac{r}{2}\right)\cos^2\varphi}\right].$$

This formula clearly shows the separate influences of dilatation or longitudinal contraction of the wire, of its flexure, the lateral slip of its individual parts, and its torsion.

I am convinced that this formula is quite useful in experiments on spiral specimens for determining the magnitude of the quantities R_0, for which - until now - only vague data had been available.

Section 5. Determination of the Displacements of Points of Solid Bodies or of Their Deformations

15. Using the notations of paragraphs Nos.2 and 3, let

- x, y, z be the coordinates of the point M of the axis of the body
- ds be the element of this axis
- ρ be its radius of curvature
- e be the angle made, on the section ω, by the prolongation of this radius with the principal axis of the ν
- $\frac{ds}{\tau}$ be the angle of two adjacent osculating planes
- δ be the characteristic of the variations by displacement

before the displacements.

$$\xi = \partial x, \quad \eta = \partial y, \quad \zeta = \partial z, \quad \varepsilon = \partial e,$$
$$X = dy\,d^2z - dz\,d^2y, \quad Y = dz\,d^2x - dx\,d^2z, \quad Z = dx\,d^2y - dy\,d^2x.$$

I found the following (restricting the calculation in this report to the case in which π_u and π_v are negligible):

$$\text{ↄ} = \frac{\delta ds}{ds} + (u \sin e + v \cos e)\frac{1}{ds}\frac{\delta ds}{\rho} + (u \cos e - v \sin e)\frac{\varepsilon}{\rho} + u\frac{dg'_0}{ds} + v\frac{dg''_0}{ds} - uv\frac{d\gamma}{ds},$$
$$\gamma = \frac{d\varepsilon}{ds} + \frac{1}{ds}\partial\frac{ds}{\tau}.$$

The last term of ∂ represents the influence of skewing on the dilatation. The term always yields zero components, and moments that generally are zero and always are small, which is the reason for the fact that this term had been neglected in paragraph No.2 and will be neglected also here. Then, eqs.(5) /952 become

$$\left\{\begin{array}{l} \frac{M_u}{E\mu} = \frac{\cos\epsilon}{ds}\cdot\delta\frac{ds}{\rho} - \sin\epsilon\frac{\epsilon}{\rho} + \frac{dg''_0}{ds}, \quad \frac{M_v}{E\mu'} = \frac{\sin\epsilon}{ds}\cdot\delta\frac{ds}{\rho} + \cos\epsilon\frac{\epsilon}{\rho} + \frac{dg'_0}{ds}, \\ \frac{P_t}{E\omega} = \frac{\delta\, ds}{ds}, \quad \frac{M_t}{G\frac{4\mu\mu'}{\mu+\mu'}} = \frac{d\epsilon}{ds} + \frac{1}{ds}\delta\frac{ds}{\tau}; \quad g'_0 = \frac{P_u}{G\omega}, \quad g''_0 = \frac{P_v}{G\omega} \end{array}\right. \tag{14}$$

By eliminating g'_0, g''_0, ϵ, this formula yields

$$\frac{\delta ds}{ds} = D, \quad \frac{1}{ds}\delta\frac{ds}{\rho} = F, \quad \delta\frac{1}{\tau} = T, \tag{15}$$

where D, F, T are polynomials not written out here and where everything is known if ξ, η, ζ are sufficiently small to exert no noticeable influence on the components and lever arms of the forces*.

On expanding the first terms in a series by differentiating, with respect to δ, the known expressions of ds, $\frac{ds}{\rho}$, and $\frac{ds}{\tau}$ in x, y, z and replacing δx, δy, δz by ξ, η, ζ, three differential equations of the first, second, and third order would be obtained between these displacements; these equations will not be given here since they contain a large number of terms. It is sufficient for me to state that I have been able to integrate them, yielding

$$\left\{\begin{array}{l} d\xi = D\,dx - dy\int\left(T dz + \frac{\rho Z}{ds^2}F\right) + dz\int\left(T dy + \frac{\rho Y}{ds^2}F\right), \\ d\eta = D\,dy - dz\int\left(T dx + \frac{\rho X}{ds^2}F\right) + dx\int\left(T dz + \frac{\rho Z}{ds^2}F\right), \\ d\zeta = D\,dz - dx\int\left(T dy + \frac{\rho Y}{ds^2}F\right) + dy\int\left(T dx + \frac{\rho X}{ds^2}F\right). \end{array}\right. \tag{16}$$

* In the opposite case, the same differential equations would be obtained, except that the second terms would then contain ξ, η, ζ and that only the difficulty of integration would be added.

17. It may be surprising to encounter in my equations a certain entirely new quantity ϵ which no one so far had ever taken into consideration and which is more or less on the same footing as the angles of contingence and of plane osculation $\frac{ds}{\rho}$ and $\frac{ds}{\tau}$. I believe that it is easy to demonstrate on an example that this angular displacement of the radius of curvature along the section must necessarily enter our analysis.

Let us imagine an elastic bar of double curvature, confined on all sides in a fixed and rigid conduit, but still rotatable about itself in view of the /953 fact that the cross section of this bar is assumed to be circular as that of the conduit. In this motion, it is assumed that the longest fibers are forcedly shortened and the shortest fibers are elongated and that also torsions are present if the rotations impressed on all cross sections had not been the same: The elasticity of the body, in all cases, would have strongly resisted such displacements of its points.

However, neither the radii of curvature nor the osculating planes of the axis will have changed in any manner whatsoever.

Consequently, the so-called resistances to bending and torsion do not depend uniquely on the variation in the angles of contingence or on the angles included between the osculating planes; these resistances depend to the same degree on another factor, namely, on the type of displacement that had taken place in the mentioned example; this is exactly, on each cross section, the angular displacement designated by me as ϵ.

It is thus obvious that it is useless to look for a solution of the problem of deformation of elastic specimens with double curvature when restricting the calculation to a consideration of only the points on their axis. It is absolutely necessary to take the events occurring outside of this axis into consideration. It seems to me that this finding explains an error by Lagrange which also had been made by Poisson* despite the fact that this error had been pointed out by Binet as early as 1814.

Section 6. Limit Conditions. General Method for Determining Unknown Reactions and Interactions in a Given System of Solid Specimens

This method consists in defining the displacements of material points of the specimens, by leaving the magnitudes, the levers, and the directions of the involved forces in an indeterminate form. Once the displacements are expressed as a function of these wanted quantities, definite conditions can be established which must be satisfied at the points of support or clamping or at the junctions of different members, or else at the points of linkage of different parts into which one and the same structural member must be subdivided because of the fact that the displacements there are expressed by different equations. In this manner, one finally will have as many equations as unknowns since, in problems of physical mechanics, obviously no indetermination whatsoever can exist.

However, these unknown forces frequently occur in an indefinite number: Reactions of the walls of the clamping device or the interactions of members that osculate over a portion of their surface; actions of other parts of the same member which have to be treated in this manner, for example, in the singular /954 case of bending of a ring or of a closed dynamometric spring all belong into this group of unknown forces. How can one enter all these small forces in the calculation? To solve this problem, I developed a method in 1837 which is by no means arbitrary and has always been successful. Let p_x, p_y, p_z be the com-

* Mécan., Second Edition, Nos. 317 and 318.

ponents, parallel to the axes, of one of the small forces and let a, b, c be the coordinates of its point of application; its moment about a straight line parallel to x, laid through the point of the axis of the body whose coordinates are x, y, z, will be $(b - y)p_x - (c - z)p_y$. However, required here are only the sums of the moments and components parallel to the three coordinates; therefore, let A_x, A_y, A_z be these sums so that the sums of the moments will become

$$B_x + A_y z - A_z y, \quad B_y + A_z x - A_x z, \quad B_z + A_x y - A_y x;$$

from which it follows that all that is necessary to know about these forces, no matter what their number might be, can be expressed for any portion of the bodies by six indeterminates at most.

The subdivision of the structural members into various parts makes it completely unnecessary to use the transcendental discontinuity formulas which Poisson had utilized*. It is only necessary to connect the axes of the individual parts of one and the same member by a common tangent or, rather, by the small angles $g' = \frac{P_u}{G\omega}$, $g'' = \frac{P_r}{G\omega}$ which will produce the slip. In specimens with double curvature, it is also necessary to give,at the limits, the value referring to the angular displacement

$$\varepsilon = \rho\left(\frac{M_u}{E\mu} - \frac{d}{ds}\cdot\frac{P_r}{G\omega}\right)\sin e - \rho\left(\frac{M_r}{E\mu'} - \frac{d}{ds}\cdot\frac{P_u}{G\omega}\right)\cos e.$$

It will be stipulated either that this displacement is zero which would apply to free ends, or that it is such that the principal axes of the section have remained stationary which is the case for clampings, and so on.

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* Mécan., No.324.